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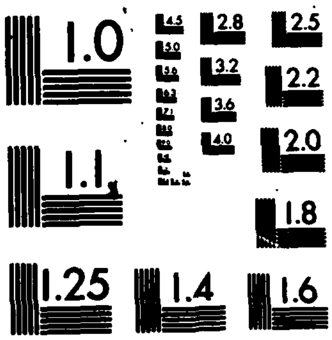

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CONTROL

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Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

Stationary Stochastic Asymmetric Control

by

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1. Introduction Let the process $X(t)$, be defined by

$$dX(t) = u(X(t))dt + dW(t)$$

where $W(t)$ is a standard Wiener process and u is a control function, and $X(0) = x$.

Let φ be even, convex, symmetric positive, exponentially bounded and strictly increasing on the positive axis. Let the average expected cost function be

$$J(x,u) = \varliminf_{T \to \infty} \frac{1}{T} \int_0^T E\,[\varphi(X(s)) + |u\,(X(s))|^{\alpha}]\,ds$$

with $\alpha > 0$.

The object is to find the control law u which minimizes J subject to $a < U < b$ where $a < 0 < b$ are real numbers. A two-dimensional version is indicated. The case $\alpha = 1$ is considered in great detail in [1]. The results for the cases $\alpha \neq 1$ are given in this paper. The complete proofs are as in [1] and reference is made to that paper for the complete proofs of the results mentioned here. In fact, it will suffice to indicate the nature of the solutions to the "asymptotic dynamic programming equation" in one-or multi-dimensions in [1], since the other arguments are as given there, or sufficiently similar so as to not be explicitly given. In one dimension it has the form

$$\lambda = \frac{f''}{2} + h(f') + \varphi ,$$

and yields the unique optimal control by the methods of [1].

II. Optimal Law

<u>Lemma 1</u>. Let $\alpha = 2n$, for $n \geq 1$ an integer. Let $a < 0 < b$.

Let $h(c) \equiv \min\limits_{a<u<b} (uc + u^{2n})$

Then

$$h(c) = \begin{cases} c\left(\frac{-c}{2n}\right)^{\frac{1}{2n-1}} \left(\frac{2n-1}{2n}\right) & \text{if } \left(\frac{-c}{2n}\right)^{\frac{1}{2n-1}} \leq b \\ 0 & \text{if } c = 0 \\ \min\left(ac + a^{2n} , bc + b^{2n}\right) & \text{otherwise} \end{cases} \tag{3}$$

Proof: The proof follows by an elementary computation, noting that the minimum is either at an interior (differentiable) point or at a boundary of u .

<u>Lemma 2</u>. If $\alpha \neq 2n$, for some integer $n \geq 1$, for $a < 0 < b$, if

$$h(c) \equiv \min_{a<u<b} (uc + |u|^{\alpha}) ,$$

then

$$h(c) = \begin{cases} c\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{|c|}{\alpha}\right)^{\frac{1}{\alpha-1}} & \text{if } c < 0 \text{ and } \left(\frac{|c|}{\alpha}\right)^{\frac{1}{\alpha-1}} \leq b \\ \min\left(0,\ ac + |a|^{\alpha},\ bc + |b|^{\alpha}\right) & \text{otherwise} \end{cases} \tag{4}$$

Proof The minimum is either at an interior differentiable point or at one of the three points 0, a, b.

Theorem 1 Assume $a < 0 < b$.

(a) If $d = 2n,\ n \geq 1$

Then there are distinct numbers $x_1 < x_2$ such that for $x_1 \leq x \leq x_2$, the "asymptotic dynamic programming equation [1], is

$$\lambda = \frac{1}{2} f'' + \left[\frac{(-f)^{2n}}{2n}\right]^{\frac{1}{2n-1}}\left(\frac{2n+1}{2n}\right) + \varphi$$

For $x > x_2$

$$\lambda = \frac{f''}{2} + af' + a^{2n} + \varphi . \tag{5}$$

The x_1, x_2, λ are found by continuity at the boundaries and by setting $f'(x_2) = f'(\infty)$ or $f'(x_1) = f'(-\infty)$.

(b) If $\alpha \neq 2n,\ \alpha \neq 1$

Then there are distinct numbers y_ℓ, $1 \le \ell \le 3$

$y_1 < 0 < y_2 < y_3$, such that the "asymptotic dynamic programming equations [1]" satisfy, if

$$x \le y_1 ,$$

$$\lambda = \frac{f''}{2} + bf' + |b|^{\alpha} + \varphi$$

and if $y_1 < x < 0,$

then

$$\lambda = \frac{f''}{2} + af' + |a|^{\alpha} + \varphi$$

and if $0 < x < y_2$

then

$$\lambda = \frac{f''}{2} + \left(\frac{(-f')^{\alpha}}{\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{\alpha+1}{\alpha}\right) + \varphi$$

and if $y_2 < x$

then

$$\lambda = \frac{f''}{2} + af' + |a|^{\alpha} + \varphi , \tag{6}$$

where the values y_1, y_2, y_3, λ are obtained by setting the solutions equal (by continuity) at y_1, y_2, y_3 and $f(y_3) = f(\infty)$ or $f(y_1) = f(-\infty)$.

Proof The expressions (5), (6) follow from (1), (2) using monotonicity and symmetry of φ, the properties of $c\,u + |u|^{\alpha}$ as a function of c, u, the consequent symmetry of f', and the uniqueness arguments in [1].

Theorem 2

a If $\alpha = 2n$, if $a < 0 < b$,

the optimal law is

$$u(X) = \begin{cases} \left(-\dfrac{f'(X)}{2n}\right)^{\frac{1}{2n-1}} & , \quad x_1 < X < x_2 \\ b & , \quad X < x_1 \\ a & , \quad X > x_2 \end{cases} \tag{7}$$

b) If $0 < \alpha \neq 1$, $\alpha \neq 2n$, then for $a < 0 < b$,

$$u(X) = \begin{cases} \left(-\dfrac{f'(X)}{\alpha}\right)^{\frac{1}{\alpha-1}} & \text{if } 0 < X < y_2 \\ b & \text{if } X < y_1 \\ a & \text{if } y_1 < X < 0 \text{ or } y_2 < X \end{cases} \tag{8}$$

<u>Proof</u> This follows immediately from Theorem 2 and the lemmas, (3) - (6). Details are as in [1] for uniqueness and optimality of u.

3. <u>Multi-dimensional case-Remarks</u>

Let $\underline{X}(t)$ be a k x 1 stochastic process with

$$d\,\underline{X}(t) = u(\underline{X}(t))dt + d\underline{W}(t) \tag{9}$$

where $\underline{u}$ is d x 1 and $\underline{W}$ is d x 1 Wiener process with independent components and $\underline{X}(0) = \underline{x}$.

Let

$$J(x,u) = \lim_{T\to\infty} \frac{1}{T} \int_0^T E\left[\varphi(\underline{X}(s)) + \|\underline{u}(\underline{X}(s))\|_p\right] ds \tag{10}$$

where φ is even, convex, positive and exponentially bounded in each argument, and where $\|u\|_p = \left(\sum_{\ell=1}^{d} |u_\ell|^p\right)^{\frac{1}{p}}$ for some $p > 0$.

It is desired to minimize J subject to

$$a_\ell \le u_\ell \le b_\ell \qquad 1 \le \ell \le d$$

where a_ℓ, b_ℓ, $1 \le \ell \le d$ are real numbers. This constraint will be written $\underline{a} \le \underline{u} \le \underline{b}$.

Let $\underline{c} = (c_1, \ldots c_d,)$ and denote

$$h(\underline{c}) \equiv \min_{\underline{a} \le \underline{u} \le \underline{b}} (\underline{c}\cdot\underline{u} + \|\underline{u}\|_p) . \tag{11}$$

The evaluation of $h(\underline{c})$ requires the evaluation of $\underline{c}\cdot\underline{u} + \|\underline{u}\|_p$ at the $2^d + 1$ points given by $u_\ell = a_\ell$ or b_ℓ, $1 \le \ell \le d$ and at the $\underline{u}$ minimizing $\underline{c}\cdot\underline{u} + \|\underline{u}\|_p$, for fixed $\underline{c}$. As $\underline{c}$ varies, R^d is cut into regions defined by the minimum $h(\underline{c})$ and the $\underline{u}$ achieving that minimum.

Let $\underline{f} = (f^{(1)}(\underline{x}), \ldots, f^{(d)}(\underline{x}))$ for $\underline{x} = (x_1, \ldots, x_d)$ such that

$$f_{ij}(\underline{x}) \equiv \frac{\partial^2 f(\underline{x})}{\partial x_i \partial x_j} \text{ exists and is continuous } 1 \le i,\ j \le d .$$

Let $f_j(\underline{x}) = \dfrac{\partial f(\underline{x})}{\partial x_j}$.

For the various regions above, it is required to solve (for minimum λ)

$$\lambda = \frac{1}{2} \Sigma f_{ij}(\underline{x}) + h(f_1(\underline{x}), \ldots f_d(\underline{x})) + \varphi(\underline{x}) . \tag{12}$$

Equating solutions at the boundary of these regions defines a finite number of vectors $\underline{x}_1 = (x_{11}, \ldots x_{1d})$, $\underline{x}_2, \ldots, \underline{x}_n$ where $n = 2^d + 1$ such that $f_j(\underline{x}_r) = f_j(\infty_r)$ where ∞_r denotes any ray in a boundary region such that at least one coordinate may be set equal to $\pm\infty$ and the point so obtained remain in the region. This determines λ also.

It follows as before that if $\underline{u}_0$ = optimal $\underline{u}$, then

$$\underline{u}_0(\underline{X}) = \text{that } \underline{u} \text{ giving } h(f_1, \ldots, f_d)$$

in any of the regions given by the vectors as edges.

Remark 2 If in addition $\|u\|_p \leq M < \infty$, and if $\underline{u}_{0p}$ = optimal $\underline{u}$ with this additional constraint, then with

$$\underline{u}_{0p} = \begin{cases} \underline{u}_0 & \text{if } \|\underline{u}_0\|_p \leq M \\ \frac{M}{R}\underline{u}_0 & \text{if } \|\underline{u}_0\|_p = R > M \end{cases} \tag{13}$$

which follows by a variational inequality. See [2] for a similar argument.

REFERENCES

1. Beneš, V, and Karatzas, I. (1980) Optimal stationary linear control of the Wiener process (unpublished manuscript).

2. Weiner, H. (1980). Constrained stochastic regulator control. (submitted).

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This document describes a which deals with a cost function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

~~The~~ one-dimensional problem ~~considered is~~ the minimization of ~~the functional~~.

$$J(x,u) = \lim_{T\to\infty} \frac{1}{T}\int_0^T E\,(\varphi(X(s)) + |u(X(s))|^{\alpha})ds$$ where $\alpha > 0$, $\alpha \neq 1$, with

$dX(t) = u(X(t))dt + dW(t)$, $X(0) = x$, and $-\infty < a < U < b < \infty$.

Generalizations to a multi-dimensional case are mentioned.

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